

Small energy Ginzburg-Landau minimizers in \mathbb{R}^3

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Abstract

We prove that a local minimizer of the Ginzburg-Landau energy in \mathbb{R}^3 satisfying the condition $\liminf_{R \rightarrow \infty} \frac{E(u; B_R)}{R \ln R} < 2\pi$ must be constant. The main tool is a new sharp η -ellipticity result for minimizers in dimension three that might be of independent interest.

1 Introduction and main results

Consider the Ginzburg-Landau equation in \mathbb{R}^N

$$-\Delta u = (1 - |u|^2)u. \quad (1.1)$$

Much effort has been devoted to classifying the solutions to (1.1) under various assumptions. In the scalar case, the famous De Giorgi conjecture states that any bounded solution satisfying $\partial_{x_N} u > 0$ must depend on one Euclidean variable only, at least when $N \leq 8$. This conjecture was proved to be true in dimension $N = 2$ by Ghoussoub and Gui [10], for $N = 3$ by Ambrosio and Cabré [2], and under the additional assumption $\lim_{x_N \rightarrow \pm\infty} u(x', x_N) = \pm 1$, for $4 \leq N \leq 8$, by Savin [20]. On the other hand, the counterexample of del Pino et al. [8] shows that indeed $N = 8$ is optimal.

Much less is known in the vector-valued case $u : \mathbb{R}^N \rightarrow \mathbb{R}^m$. In that case the monotony hypothesis no longer makes sense. On the other hand, the class of locally minimizing solutions in the sense of De Giorgi — i.e., solutions that are minimizing for their own boundary values on every ball — come up naturally as blow-up limits of minimizers of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \int \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2,$$

as ε goes to zero. In fact, monotone scalar solutions of (1.1) also have a certain local minimality property (see [1, Thm 4.4]).

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When $N = m = 2$ it follows from the results of Sandier [18] and Mironescu [15] that every local minimizer is either constant or equal to $U(x) := f(|x|)^{\frac{x}{|x|}}$, up to rotations and translations, where f is the unique solution of the corresponding ODE. When $N = m = 3$ a similar classification was proved by Millot and Pisante [14] under the assumption

$$\limsup_{R \rightarrow \infty} \frac{E(u; B_R)}{R} < \infty \quad (\text{here } E(u; B_R) = E_1(u; B_R)).$$

Under additional assumptions, Pisante [16] extended the result to $N = m \geq 4$. Note that in these cases the existence of a non-constant local minimizer is a nontrivial fact.

We are interested in the case $N = 3, m = 2$. In this case, it is easy to deduce from the local minimality property of U that $V(x, z) = U(x)$ is a local minimizer. We conjecture that it is the only non-constant one, up to the obvious symmetries of the problem. As a first step in this direction we prove:

Theorem 1.1. *If $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a local minimizer of the Ginzburg-Landau energy such that*

$$\liminf_{R \rightarrow \infty} \frac{E(u; B_R)}{R \ln R} < 2\pi,$$

then u is constant.

Note that the constant 2π is optimal since

$$\lim_{R \rightarrow \infty} \frac{E(V; B_R)}{R \ln R} = 2\pi.$$

A different assumption was considered by Farina [9] who proved that a local minimizer $u : \mathbb{R}^N \rightarrow \mathbb{R}^2$ with $N = 3$ or 4 , is constant provided $\lim_{|x| \rightarrow \infty} |u(x)| = 1$.

Theorem 1.1 is an easy consequence of the following sharp η -ellipticity type result for minimizers in dimension three:

Theorem 1.2. *For every $\gamma \in (0, 2\pi)$ and $\lambda > 0$ there exists $\varepsilon_1(\gamma, \lambda)$ such that for every u_ε which is a minimizer of E_ε on B_1 with $\varepsilon \leq \varepsilon_1(\gamma, \lambda)$ satisfying $E_\varepsilon(u; B_1) \leq \gamma |\ln \varepsilon|$ there holds*

$$||u_\varepsilon(0)| - 1| \leq \lambda. \tag{1.2}$$

Such a result first appeared in the work of Rivière [17] for the case of minimizers in dimension three. There were subsequent generalizations by Lin-Rivière [12, 13] and Bethuel-Brezis-Orlandi [4, 5]. The result in [5] is the most general, covering the case of solutions (not necessary minimizers) of the Ginzburg-Landau equation in any dimension.

All these results establish the existence of a constant $\gamma > 0$ for which the result is true, but no explicit bound is given. We are able to give the optimal bound, but only for minimizers in dimension three. Working with minimizers allows us to apply a construction of an appropriate test function. This is done in Proposition 1.3 below, which plays an important role in the proof of Theorem 1.2 (see Section 2 for notation).

Proposition 1.3. *Let u be a minimizer for E on $B_R \subset \mathbb{R}^3$, for its boundary values on S_R . If*

$$E^{(T)}(u; S_R) \leq \gamma \ln R, \tag{1.3}$$

with $\gamma < 2\pi$ and $R > r_0(\gamma)$ then there exist $\alpha = \alpha(\gamma) \in (0, 1)$ and a universal $\sigma \in (0, 1)$ such that

$$E(u; B_R) \leq \sigma R E^{(T)}(u; S_R) + C_\gamma R^\alpha \ln R. \tag{1.4}$$

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2 Notation and some preliminary results

We begin by introducing some notation. By $B_R(a)$ we denote a ball in \mathbb{R}^N , $N \geq 2$ (usually for $N = 3$) and then $S_R(a) = \partial B_R(a)$. Specifically in dimension two, we denote by $D_R(a)$ a disc and by $C_R(a) = \partial D_R(a)$ its boundary. In case $a = 0$ we denote for short:

$$B_R = B_R(0), \quad S_R = \partial B_R(0), \quad D_R = D_R(0), \quad C_R = \partial D_R(0).$$

For a set D in \mathbb{R}^N , $N \geq 2$, $\varepsilon > 0$ and $u \in H^1(D; \mathbb{R}^2)$ we denote:

$$E_\varepsilon(u; D) = \int_D \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2.$$

For $\tilde{D} \subseteq S_R(a) \subset \mathbb{R}^N$, $N \geq 2$, and $u \in H^1(\tilde{D}; \mathbb{R}^2)$ we denote

$$E_\varepsilon^{(T)}(u; \tilde{D}) = \int_{\tilde{D}} \frac{1}{2} |\nabla_T u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2.$$

When $\varepsilon = 1$ we denote for short $E(u; D) = E_1(u; D)$ and $E^{(T)}(u; \tilde{D}) = E_1^{(T)}(u; \tilde{D})$.

We denote by $\tilde{D}_\rho^R(a)$ a spherical disc on the sphere $S_R = S_R(0) \subset \mathbb{R}^3$, with center at a and radius ρ (using the geodesic distance) and by $\tilde{C}_\rho^R(a) = \partial \tilde{D}_\rho^R(a)$ its boundary. In case there is no risk of confusion we shall omit the superscript R . We often identify \mathbb{R}^2 -valued maps with \mathbb{C} -valued maps.

We recall the following basic estimates valid for an arbitrary (vector valued) solution of the Ginzburg-Landau equation (1.1) in dimension $N \geq 2$. The first is a L^∞ -bound for u and its gradient ([5, Lemma III.2]).

Lemma 2.1. *Assume u satisfies (1.1) in $B_1 \subset \mathbb{R}^N$. Then, there is a constant $K = K(N) > 0$ such that*

$$\|u\|_{L^\infty(B_{1/2})} \leq K, \tag{2.1}$$

and

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq K. \tag{2.2}$$

Sketch of proof. For the proof of (2.1) we argue as in Brezis [7, Remark 3] to get, using Kato's inequality, that $\varphi = (|u|^2 - 1)^+$ satisfies

$$\Delta \varphi \geq \varphi^2 \quad \text{in } B_R. \tag{2.3}$$

The result then follows by the Keller-Osserman theory (see [6] and the references therein).

Once (2.1) is established, (2.2) follows by standard elliptic estimates. \square

The second is a version of the monotonicity formula ([5, Corollary II.1]):

Lemma 2.2. *Let u be a solution of (1.1) in $B_R \subset \mathbb{R}^N$. Then, the function $r \mapsto E(u; B_r)/r^{N-2}$ is nondecreasing in $(0, R)$.*

Another useful result is the following estimate for harmonic functions in balls.

Lemma 2.3. *Let $w \in C^1(\overline{B_R})$ be a harmonic function in a ball $B_R \subset \mathbb{R}^N$. Then*

$$\int_{B_R} |\nabla w|^2 \leq \frac{R}{N-1} \int_{\partial B_R} |\nabla_T w|^2. \quad (2.4)$$

Proof. First note that since $|\nabla w|^2$ is subharmonic we have

$$\int_{B_R} |\nabla w|^2 \leq \int_0^R \left(\frac{r^{N-1}}{R^{N-1}} \int_{\partial B_R} |\nabla w|^2 \right) dr = \frac{R}{N} \int_{\partial B_R} |\nabla w|^2,$$

i.e.,

$$N \int_{B_R} |\nabla w|^2 \leq R \int_{\partial B_R} \left(|\nabla_T w|^2 + \left| \frac{\partial w}{\partial n} \right|^2 \right). \quad (2.5)$$

On the other hand, Pohozaev identity gives

$$(N-2) \int_{B_R} |\nabla w|^2 = R \int_{\partial B_R} \left(|\nabla_T w|^2 - \left| \frac{\partial w}{\partial n} \right|^2 \right). \quad (2.6)$$

Adding (2.5) with (2.6) yields (2.4). \square

3 Proof of Proposition 1.3

The next lemma deals with a Ginzburg-Landau minimization problem on a spherical disc. The proof requires simple modification of the methods developed for the case of the usual Ginzburg-Landau energy in the plane with zero degree boundary condition (see [3]).

Lemma 3.1. *Let $u \in H^1(\tilde{D}_\rho^R(a), \mathbb{R}^2)$ with*

$$\rho/R < 1/10 \quad (3.1)$$

and

$$E^{(T)}(u; \tilde{C}_\rho^R(a)) \leq c_1/\rho. \quad (3.2)$$

Then, if $\rho \geq R_0(c_1)$ we have

$$||u| - 1| \leq 1/8 \quad \text{on } \tilde{C}_\rho^R(a). \quad (3.3)$$

If we further assume that

$$\deg(u, \tilde{C}_\rho^R(a)) = 0, \quad (3.4)$$

then for $\rho \geq R_1(c_1) \geq R_0(c_1)$, any minimizer v of the energy $E(\cdot; \tilde{D}_\rho^R(a))$ for the boundary data u on $\tilde{C}_\rho^R(a)$, satisfies:

$$|1 - |v|| \leq 1/8 \quad \text{on } \tilde{D}_\rho^R(a). \quad (3.5)$$

Proof. The proof of (3.3) is easy and standard (see [11, Lemma 1]). Indeed, assume that for some point $x_0 \in \tilde{C}_\rho = \tilde{C}_\rho^R(a)$ we have

$$||u(x_0)| - 1| > 1/8. \quad (3.6)$$

Then, for any $x_1 \in \tilde{C}_\rho$ we have, using Hölder inequality on the arc of circle $A(x_0, x_1) \subset \tilde{C}_\rho$ and (3.2):

$$|u(x_0) - u(x_1)| \leq \left(2E^{(T)}(u; \tilde{C}_\rho)\right)^{1/2} |A(x_0, x_1)|^{1/2} \leq \left(\frac{2c_1}{\rho}\right)^{1/2} |A(x_0, x_1)|^{1/2}.$$

It follows, using (3.6), that there exists $\lambda > 0$ such that

$$||u| - 1| \geq 1/16 \quad \text{on } \{|x - x_0| < \lambda\rho/c_1\} \cap \tilde{C}_\rho. \quad (3.7)$$

Clearly, (3.7) implies, for some positive constant c_2 , that

$$\int_{\tilde{C}_\rho} (1 - |u|^2)^2 \geq c_2\rho,$$

which contradicts (3.2) for ρ large enough.

For the proof of (3.5) it is convenient to treat an equivalent problem for a certain weighted Ginzburg-Landau energy on a planar disc. For that matter we will perform a change of variables in several steps. Without loss of generality we may assume that $a = (0, \dots, 0, -R)$, the south pole of S_R . Denoting by $\mathcal{S} : S_1 \rightarrow \mathbb{R}^2$ the standard stereographic projection we define a function $\tilde{u} : D_{\tan(\rho/2R)} \rightarrow \mathbb{R}^2$ by $\tilde{u}(x) = u(R\mathcal{S}^{-1}x)$. We have

$$E(u; \tilde{D}_\rho^R(a)) = \int_{D_{\tan(\rho/2R)}} \left\{ \frac{1}{2} |\nabla \tilde{u}|^2 + \frac{R^2(1 - |\tilde{u}|^2)^2}{(1 + |x|^2)^2} \right\} dx.$$

A final rescaling, setting $U(y) = \tilde{u}(\tan(\rho/2R)y)$, yields $U : D_1 \rightarrow \mathbb{R}^2$ satisfying

$$F_\varepsilon(U; D_1) = E(u; \tilde{D}_\rho^R(a))$$

with

$$F_\varepsilon(U; D_1) := \int_{D_1} \left\{ \frac{1}{2} |\nabla U|^2 + \frac{p(y)}{4\varepsilon^2} (1 - |U|^2)^2 \right\} dy,$$

where

$$\varepsilon = 1/(2R \tan(\rho/2R)) \quad \text{and} \quad p(y) = 1/(1 + \tan^2(\rho/2R)|y|^2)^2.$$

Note that by (3.1) $p(y)$ is bounded between two positive constants, and all its derivatives are uniformly bounded as well. Moreover, by (3.2) we have

$$\int_{\partial D_1} \left\{ \frac{1}{2} |\nabla U|^2 + \frac{p(y)}{4\varepsilon^2} (1 - |U|^2)^2 \right\} \leq \tilde{c}_1,$$

where \tilde{c}_1 depends on c_1 only.

The proof of (3.5) follows by a simple modification of the arguments in [3, Thm 2] (in particular the proof of (95) there), see also [11, Lemma 2]; the fact that here we deal with a weighted Ginzburg-Landau energy causes no difficulty since the weight is smooth, as explained above. □

The next lemma is concerned with an extension problem in a cylinder. It will be useful for a similar problem on a spherical cylinder in the course of the proof of Proposition 1.3.

Lemma 3.2. *Let $u, v \in H^1(D_R, \mathbb{R}^2)$ be such that*

$$u = v \text{ on } C_R, \quad (3.8)$$

$$|1 - |v|| \leq 1/8 \text{ and } |u| \leq K \text{ on } \overline{D}_R. \quad (3.9)$$

Then, for any $H > 0$ there exists $U \in H^1(D_R \times (0, H))$ satisfying

$$U(x, H) = u(x), \quad x \in D_R \quad (3.10)$$

$$U(x, 0) = v(x), \quad x \in D_R \quad (3.11)$$

$$U(x, z) = u(x) = v(x), \quad x \in \partial D_R, z \in (0, H) \quad (3.12)$$

and

$$E(U; D_R \times (0, H)) \leq C(H + R^2/H) (E(u; D_R) + E(v; D_R)). \quad (3.13)$$

Proof. We first extend v to the cylinder $\overline{D}_R \times [0, H]$ by letting

$$V(x, z) = v(x), \quad x \in \overline{D}_R, z \in [0, H]. \quad (3.14)$$

Define the cone Γ by

$$\Gamma = \{(x, z); 0 < |x| < R, (H/R)|x| < z < H\}. \quad (3.15)$$

Let w be defined in \overline{D}_R by $w = u/v$ and then extend it to $\overline{D}_R \times [0, H]$ by

$$W(x, z) = \begin{cases} 1 & (x, z) \in \overline{D}_R \times [0, H] \setminus \Gamma \\ w(Hx/z) & (x, z) \in \Gamma \end{cases}. \quad (3.16)$$

Finally we set

$$U = VW \quad \text{in } \overline{D}_R \times [0, H]. \quad (3.17)$$

Clearly,

$$E(W; \overline{D}_R \times [0, H]) = E(W; \Gamma) = \int_0^H dz \int_{D_{Rz/H} \times \{z\}} \frac{1}{2} |\nabla W|^2 + \frac{1}{4} (1 - |W|^2)^2. \quad (3.18)$$

First note that

$$\int_{D_{Rz/H} \times \{z\}} |\nabla_x W|^2 = \int_{D_R} |\nabla w|^2, \quad (3.19)$$

while

$$\begin{aligned} \int_{D_{Rz/H} \times \{z\}} \left| \frac{\partial W}{\partial z} \right|^2 &= \int_{D_{Rz/H}} |\nabla w(Hx/z) \cdot (Hx/z^2)|^2 \\ &\leq \frac{1}{H^2} \int_{D_R} |\nabla w(y)|^2 |y|^2 dy \leq \left(\frac{R}{H} \right)^2 \int_{D_R} |\nabla w|^2. \end{aligned} \quad (3.20)$$

Integrating (3.19)–(3.20) over $z \in (0, H)$ yields

$$\int_{D_R \times (0, H)} |\nabla W|^2 \leq (H + R^2/H) \int_{D_R} |\nabla w|^2. \quad (3.21)$$

Since by (3.9)

$$|\nabla w| \leq C(|\nabla u| + |\nabla v|), \quad (3.22)$$

we deduce from (3.21) that

$$\int_{D_R \times (0, H)} |\nabla W|^2 \leq C(H + R^2/H) \left(\int_{D_R} |\nabla u|^2 + \int_{D_R} |\nabla v|^2 \right). \quad (3.23)$$

Next we turn to estimate the second term in the energy. We have

$$\begin{aligned} \int_{D_{Rz/H}} (|W(x, z)|^2 - 1)^2 dx &= \int_{D_{Rz/H}} (|w(Hx/z)|^2 - 1)^2 dx \\ &= (z/H)^2 \int_{D_R} (|w(y)|^2 - 1)^2 dy, \end{aligned} \quad (3.24)$$

so integrating (3.24) for $z \in (0, H)$ gives

$$\int_{\Gamma} (|W|^2 - 1)^2 = (H/3) \int_{D_R} (|w(y)|^2 - 1)^2 dy. \quad (3.25)$$

Noting that

$$||w|^2 - 1| = \frac{||u|^2 - |v|^2|}{|v|^2} \leq (8/7)^2 (||u|^2 - 1| + ||v|^2 - 1|), \quad (3.26)$$

we conclude from (3.25) that

$$\int_{\Gamma} (|W|^2 - 1)^2 \leq CH \left(\int_{D_R} (|u|^2 - 1)^2 + \int_{D_R} (|v|^2 - 1)^2 \right). \quad (3.27)$$

We also clearly have

$$\int_{D_R \times (0, H)} |\nabla V|^2 = H \int_{D_R} |\nabla v|^2, \quad (3.28)$$

$$\int_{D_R \times (0, H)} (1 - |V|^2)^2 = H \int_{D_R} (1 - |v|^2)^2. \quad (3.29)$$

Similarly to (3.22) we have $|\nabla U| \leq c(|\nabla V| + |\nabla W|)$. Hence, from (3.23) and (3.28) we get that

$$\int_{D_R \times (0, H)} |\nabla U|^2 \leq C(H + R^2/H) \left(\int_{D_R} |\nabla u|^2 + \int_{D_R} |\nabla v|^2 \right). \quad (3.30)$$

By (3.17), (3.9) and (3.14) we have

$$||U|^2 - 1| \leq ||V|^2 |W|^2 - |V|^2| + ||V|^2 - 1| \leq (9/8)^2 ||W|^2 - 1| + ||V|^2 - 1|,$$

and applying (3.29) and (3.27) yields

$$\int_{D_R \times (0, H)} (1 - |U|^2)^2 \leq CH \left(\int_{D_R} (1 - |v|^2)^2 + \int_{D_R} (1 - |u|^2)^2 \right). \quad (3.31)$$

Clearly (3.13) follows from (3.30)–(3.31). \square

Proof of Proposition 1.3. The proof is divided to several steps.

Step 1: Locating the “bad discs” on S_R and choosing α .

The following result should seem plausible to specialists in the field; we state it as a Lemma and prove it in the appendix.

Lemma 3.3. *Given $\gamma < 2\pi$ and $\Lambda > 0$ there exists $R_0 > 0$ such that if $R > R_0$ and $u : S_R \rightarrow \mathbb{R}^2$ satisfies*

$$E^{(T)}(u; S_R) \leq \gamma \ln R,$$

then there exist $\alpha \in (0, 1)$ and spherical discs $\tilde{D}_i = \tilde{D}_{r_i}^R(a_i)$, $i = 1, \dots, k$, such that

$$|u| > 7/8 \text{ on } S_R \setminus \bigcup_{i=1}^k \tilde{D}_i, \quad (3.32)$$

$$\sum_{i=1}^k r_i \leq R^\alpha, \quad (3.33)$$

$$E^{(T)}(u, \partial \tilde{D}_i) \leq \frac{2\pi}{r_i}, \quad i = 1, \dots, k, \quad (3.34)$$

$$\deg(u, \partial \tilde{D}_i) = 0, \quad i = 1, \dots, k, \quad (3.35)$$

$$r_i \geq \Lambda, \quad i = 1, \dots, k. \quad (3.36)$$

Applying Lemma 3.3 with $\Lambda = R_1(2\pi)$ (see Lemma 3.1) yields a collection $\{\tilde{D}_{r_i}^R(a_i)\}_{i=1}^k$ satisfying (3.32)–(3.36).

Step 2: Extension to a map in $B_R \setminus B_{R-R^\alpha}$ with a phase on S_{R-R^α} .

For each $i = 1, \dots, k$ we apply Lemma 3.1 with $c_1 = 2\pi$ to find a map v_i defined in $\tilde{D}_i = \tilde{D}_{r_i}^R(a_i)$ satisfying:

$$v_i = u \text{ on } \partial \tilde{D}_i, \quad (3.37)$$

$$||v_i| - 1| \leq 1/8 \text{ on } \tilde{D}_i, \quad (3.38)$$

$$E^{(T)}(v_i; \tilde{D}_i) \leq E^{(T)}(u; \tilde{D}_i). \quad (3.39)$$

Note that for any spherical cylinder of the form

$$\mathcal{C} = \left\{ (r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, \cos \varphi) ; r \in (R - H, R), \varphi \in [0, \rho/R], \theta \in [0, 2\pi) \right\}$$

there is a C^1 diffeomorphism $\Psi : D_\rho \times (0, H) \rightarrow \mathcal{C}$ given by

$$\Psi(y_1, y_2, y_3) = \left(\frac{y_3 + R - H}{R} \right) (y_1, y_2, \sqrt{R^2 - y_1^2 - y_2^2}).$$

A direct computation yields that

$$D\Psi = \text{Id} + O(\rho/R) + O(H/R) \text{ and } D(\Psi^{-1}) = \text{Id} + O(\rho/R) + O(H/R). \quad (3.40)$$

Consider each spherical cylinder

$$\mathcal{C}_i = \left\{ ry ; y \in \tilde{D}_i, r \in \left(\frac{R - R^\alpha}{R}, 1 \right) \right\}, \quad i = 1, \dots, k.$$

By applying a rotation (sending a_i to \mathcal{N}) and then the map Ψ^{-1} we find a regular cylinder $D_{r_i} \times (0, R^\alpha)$ on which we perform the construction of Lemma 3.2. Going back to \mathcal{C}_i , we get, taking into account (3.40), a map $U_i \in H^1(\mathcal{C}_i, \mathbb{R}^2)$ satisfying:

$$U_i(x) = u(x), \quad x \in \tilde{D}_i, \quad (3.41)$$

$$U_i(x) = v_i(Rx/(R - R^\alpha)), \quad x \in ((R - R^\alpha)/R)\tilde{D}_i \subset S_{R-R^\alpha}, \quad (3.42)$$

$$U_i(x) = u(Rx/|x|) = v_i(Rx/|x|), \quad \frac{Rx}{|x|} \in \partial\tilde{D}_i, \quad (3.43)$$

and

$$E(U_i; \mathcal{C}_i) \leq C(R^\alpha + r_i^2/R^\alpha) E^{(T)}(u; \tilde{D}_i) \leq CR^\alpha E^{(T)}(u; \tilde{D}_i). \quad (3.44)$$

Denoting by $\mathcal{P}_R(x) = Rx/|x|$ the radial projection on S_R , we finally define U in $B_R \setminus B_{R-R^\alpha}$ by

$$U(x) = \begin{cases} U_i(x) & \text{if } \mathcal{P}_R(x) \in \tilde{D}_i \text{ for some } i \\ u(\mathcal{P}_R(x)) & \text{otherwise} \end{cases}. \quad (3.45)$$

We clearly have

$$\begin{aligned} U(x) &= u(x) \text{ on } S_R \\ V(x) &:= U|_{S_{R-R^\alpha}}(x) \text{ satisfies } 7/8 \leq |V(x)| \leq K \text{ on } S_{R-R^\alpha}. \end{aligned} \quad (3.46)$$

Moreover,

$$E^{(T)}(V; S_{R-R^\alpha}) \leq E^{(T)}(u; S_R), \quad (3.47)$$

$$E(U; B_R \setminus B_{R-R^\alpha}) \leq CR^\alpha \ln R. \quad (3.48)$$

Indeed, (3.47) follows by summing up the contribution of each \tilde{D}_i in (3.39) and using (3.45). The inequality (3.48) follows from (3.44)–(3.45) and (1.3).

Step 3: Extension in B_{R-R^α} .

On S_{R-R^α} we may write

$$U(x) = \tilde{\rho}(x)e^{i\tilde{\varphi}(x)}, \quad (3.49)$$

with $7/8 \leq \tilde{\rho} \leq K$. We first extend to $U(x) = \rho(x)e^{i\varphi(x)}$ in $A(R) := B_{R-R^\alpha} \setminus \overline{B_{R-2R^\alpha}}$ by setting:

$$\rho(x) = \frac{r - (R - 2R^\alpha)}{R^\alpha} \tilde{\rho}((R - R^\alpha)x/|x|) + \frac{(R - R^\alpha) - r}{R^\alpha} \quad (3.50)$$

$$\varphi(x) = \varphi((R - R^\alpha)x/|x|), \quad (3.51)$$

where $r = |x| \in (R - 2R^\alpha, R - R^\alpha)$. A direct computation gives:

$$\int_{A(R)} \left| \frac{\partial \rho}{\partial r} \right|^2 \leq \frac{C}{R^\alpha} \ln R, \quad (3.52)$$

$$\int_{A(R)} |\nabla_T \rho|^2 \leq CR^\alpha \ln R, \quad (3.53)$$

$$\int_{A(R)} (1 - \rho^2)^2 \leq CR^\alpha \ln R, \quad (3.54)$$

$$\int_{A(R)} |\nabla \varphi|^2 \leq CR^\alpha \ln R. \quad (3.55)$$

From (3.52)–(3.55) it follows that

$$E(U; A(R)) \leq CR^\alpha \ln R. \quad (3.56)$$

Moreover, by (3.51) and (3.45)–(3.46) we have

$$\int_{S_{R-2R^\alpha}} |\nabla \varphi|^2 = \int_{S_{R-R^\alpha}} |\nabla \varphi|^2 \leq 2(8/7)^2 E^{(T)}(u; S_R). \quad (3.57)$$

Finally, denoting by Φ the harmonic extension of φ to B_{R-2R^α} , we set

$$U(x) = e^{i\Phi(x)} \quad \text{in } B_{R-2R^\alpha}.$$

Combining (2.4), (3.57), (3.48) and (3.56) we obtain

$$E(U; B_R) \leq (1/2)(8/7)^2(R - 2R^\alpha)E^{(T)}(u; S_R) + CR^\alpha \ln R, \quad (3.58)$$

that clearly implies (1.4) for R large, since $E(u; B_R) \leq E(U; B_R)$. \square

4 Proof of the main results

We begin with the proof of Theorem 1.2. It will be more convenient to prove instead the theorem under the following equivalent formulation, obtained by rescaling:

Theorem 4.1. *For every $\gamma \in (0, 2\pi)$ and $\lambda > 0$ there exists $R_1(\gamma, \lambda)$ such that for every u which is a minimizer of E on B_R , with $R \geq R_1(\gamma, \lambda)$, satisfying $E(u; B_R) \leq \gamma R \ln R$ there holds*

$$||u(0)| - 1| \leq \lambda. \quad (4.1)$$

Proof of Theorem 4.1. We will use the shorthand $E(R)$ for $E(u; B_R)$ and also

$$\begin{aligned} E'(R) &:= \frac{d}{dR} E(R) = \int_{S_R} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2, \\ e_T(R) &:= E^{(T)}(u; S_R) = \int_{S_R} \frac{1}{2} |\nabla_T u|^2 + \frac{1}{4} (1 - |u|^2)^2. \end{aligned}$$

The value of $R_1 = R_1(\gamma, \lambda)$ will be determined later. For the moment we will assume its value is known and in the course of the proof we shall obtain some constraints that will allow us to determine its value definitively. We first fix ε satisfying

$$\gamma + 3\varepsilon < 2\pi. \quad (4.2)$$

Next we take any $\beta \in (0, 1)$, e.g., $\beta = 1/2$. Since

$$\int_{R^\beta}^R E'(r) dr \leq E(R) < \gamma R \ln R = \gamma \int_{R^\beta}^R (\ln r + 1) dr + \gamma R^\beta \ln(R^\beta),$$

there exists $\rho_1 \in [R^\beta, R]$ such that

$$E'(\rho_1) \leq \gamma \left(\ln \rho_1 + 1 + \frac{R^\beta \ln(R^\beta)}{R - R^\beta} \right) \leq (\gamma + \varepsilon) \ln \rho_1, \quad (4.3)$$

if R_1 is chosen large enough. Since $e_T(\rho_1) \leq E'(\rho_1) \leq (\gamma + \varepsilon) \ln \rho_1$, we may apply Proposition 1.3 (with $\gamma + \varepsilon$ playing the role of γ) to get

$$E(\rho_1) \leq \sigma \rho_1 e_T(\rho_1) + C \rho_1^\alpha \ln \rho_1 \leq \sigma(\gamma + 2\varepsilon) \rho_1 \ln \rho_1, \quad (4.4)$$

provided R_1 is large enough. Next we fix $\delta \in (0, 1)$ satisfying

$$\gamma_0 := \frac{(\gamma + 3\varepsilon)}{\delta} < 2\pi. \quad (4.5)$$

By (4.4) we have, for every $r \in [\delta \rho_1, \rho_1]$,

$$E(r) \leq E(\rho_1) \leq \frac{\sigma(\gamma + 2\varepsilon)}{\delta} \delta \rho_1 \ln \rho_1 < (\sigma \gamma_0) \delta \rho_1 \ln(\delta \rho_1) \leq (\sigma \gamma_0) r \ln r, \quad (4.6)$$

again, if R_1 is chosen large enough. Let $M > 0$ denote a constant such that for $R \geq M$ all the inequalities, (4.3), (4.4) and (4.6) hold true. Set

$$\rho_2 := \inf\{R \in [\tilde{R}_0, \rho_1] : E(s) < (\sigma \gamma_0) s \ln s, \forall s \in [R, \rho_1]\}, \quad (4.7)$$

where $\tilde{R}_0 = \tilde{R}_0(\gamma_0, \sigma) \geq r_0(\gamma_0)$ (see Proposition 1.3) will be determined below. Thanks to (4.6) we have $\rho_2 \leq \delta \rho_1$. For $r \in [\rho_2, \rho_1]$ we have, either

$$E'(r) \geq \frac{E(r)}{\sigma r}, \quad (4.8)$$

or

$$E'(r) < \frac{E(r)}{\sigma r}. \quad (4.9)$$

In the latter case, when (4.9) holds, we have by (4.7),

$$e_T(r) \leq E'(r) < \frac{E(r)}{\sigma r} \leq \gamma_0 \ln r.$$

Applying Proposition 1.3 yields

$$E'(r) \geq e_T(r) \geq \frac{E(r) - C r^\alpha \ln r}{\sigma r}. \quad (4.10)$$

The first possibility (4.8) clearly implies (4.10). Hence, for every $r \in [\rho_2, \rho_1]$, (4.10) holds. We rewrite (4.10) as

$$(r^{-1/\sigma} E(r))' \geq -\frac{C}{\sigma} r^{\alpha-1-1/\sigma} \ln r. \quad (4.11)$$

Integrating (4.11) for $r \in [\rho_2, \rho_1]$ gives

$$\frac{E(\rho_1)}{\rho_1^{1/\sigma}} - \frac{E(\rho_2)}{\rho_2^{1/\sigma}} \geq -\frac{C}{\sigma} \left(\frac{\rho_1^{\alpha-1/\sigma}}{\alpha-1/\sigma} \ln \rho_1 - \frac{\rho_1^{\alpha-1/\sigma}}{(\alpha-1/\sigma)^2} - \frac{\rho_2^{\alpha-1/\sigma}}{\alpha-1/\sigma} \ln \rho_2 + \frac{\rho_2^{\alpha-1/\sigma}}{(\alpha-1/\sigma)^2} \right), \quad (4.12)$$

whence by (4.6),

$$\begin{aligned} E(\rho_2) &\leq \left(\frac{\rho_2}{\rho_1}\right)^{1/\sigma} E(\rho_1) + C(\alpha, \sigma) \rho_2^\alpha \ln \rho_2 \leq \left(\frac{\rho_2}{\rho_1}\right)^{1/\sigma} \sigma \gamma_0 (\delta \rho_1) \ln(\delta \rho_1) + C(\alpha, \sigma) \rho_2^\alpha \ln \rho_2 \\ &= (\sigma \gamma_0) \delta^{1/\sigma} \rho_2^{1/\sigma} (\delta \rho_1)^{1-1/\sigma} \ln(\delta \rho_1) + C(\alpha, \sigma) \rho_2^\alpha \ln \rho_2. \end{aligned} \quad (4.13)$$

Using $\rho_2 \leq \delta \rho_1$ and the fact that the function $t \mapsto (\ln t)t^{1-1/\sigma}$ is decreasing for $t \geq c_0$, we obtain from (4.13) that

$$\begin{aligned} E(\rho_2) &\leq (\sigma\gamma_0)\delta^{1/\sigma}\rho_2^{1/\sigma} \cdot \rho_2^{1-1/\sigma} \ln \rho_2 + C(\alpha, \sigma)\rho_2^\alpha \ln \rho_2 \\ &= (\sigma\gamma_0)\delta^{1/\sigma}\rho_2 \ln \rho_2 + C(\alpha, \sigma)\rho_2^\alpha \ln \rho_2. \end{aligned} \quad (4.14)$$

Let r_1 denote the value of $\rho_2 \neq 1$ for which equality holds between the right hand sides of (4.14) and (4.7) (for $s = \rho_2$). That is, r_1 satisfies the equality

$$(\sigma\gamma_0)r_1 = (\sigma\gamma_0)\delta^{1/\sigma}r_1 + C(\alpha, \sigma)r_1^\alpha. \quad (4.15)$$

A simple computation gives

$$r_1 = \left(\frac{C(\alpha, \sigma)}{\sigma\gamma_0(1 - \delta^{1/\sigma})} \right)^{\frac{1}{1-\alpha}}, \quad (4.16)$$

and we may indeed assume that $r_1 > 1$ by replacing, if necessary, $C(\alpha, \sigma)$ by a larger constant. Note that

$$(\sigma\gamma_0)\rho \ln \rho > (\sigma\gamma_0)\delta^{1/\sigma}\rho \ln \rho + C(\alpha, \sigma)\rho^\alpha \ln \rho \quad \text{for } \rho > r_1. \quad (4.17)$$

Next we claim that if we take in (4.7) $\tilde{R}_0 = \tilde{R}_0(\gamma)$ satisfying

$$\tilde{R}_0 \geq \max\{r_0(\gamma_0), r_1, M\}, \quad (4.18)$$

then necessarily

$$\rho_2 = \tilde{R}_0. \quad (4.19)$$

Indeed, otherwise $\rho_2 > \tilde{R}_0$ and by (4.7) and (4.17) we must have

$$E(\rho_2) = (\sigma\gamma_0)\rho_2 \ln \rho_2 > (\sigma\gamma_0)\delta^{1/\sigma}\rho_2 \ln \rho_2 + C(\alpha, \sigma)\rho_2^\alpha \ln \rho_2, \quad (4.20)$$

which contradicts the bound in (4.14).

In view of (4.19) we know that (4.11) holds for all $r \in [\tilde{R}_0, \rho_1]$, and integration over this interval yields, as in (4.12)–(4.13),

$$\begin{aligned} E(\tilde{R}_0) &\leq \sigma\gamma_0\delta\rho_1^{1-1/\sigma}\tilde{R}_0^{1/\sigma} \ln(\delta\rho_1) + C(\alpha, \sigma)\tilde{R}_0^\alpha \ln \tilde{R}_0 \\ &= \sigma\gamma_0\delta\tilde{R}_0^\alpha \left(\tilde{R}_0^{1/\sigma-\alpha}/\rho_1^{1/\sigma-1} \right) \ln(\delta\rho_1) + C(\alpha, \sigma)\tilde{R}_0^\alpha \ln \tilde{R}_0. \end{aligned} \quad (4.21)$$

Let $\tilde{R}_1 = \tilde{R}_1(\tilde{R}_0)$ be determined by the equation

$$\tilde{R}_0^{1/\sigma-\alpha} = \tilde{R}_1^{\beta(1/\sigma-1)} / \ln(\tilde{R}_1^\beta) \leq \rho_1^{1/\sigma-1} / \ln \rho_1. \quad (4.22)$$

It follows from (4.21) that for every $R_1 \geq \tilde{R}_1(\tilde{R}_0)$, where \tilde{R}_0 satisfies (4.18), we have (under the assumption of the Theorem, i.e., $E(R) \leq \gamma R \ln R$ for some $R \geq R_1$):

$$E(\tilde{R}_0) \leq \sigma\gamma_0\delta\tilde{R}_0^\alpha + C(\alpha, \sigma)\tilde{R}_0^\alpha \ln \tilde{R}_0 \leq \tilde{C}\tilde{R}_0^\alpha \ln \tilde{R}_0, \quad (4.23)$$

for some \tilde{C} (which actually depends only on γ).

Finally we turn to the proof of (4.1). It is clearly enough to consider $\lambda < 2K$ (where K is given by Lemma 2.1). Looking for contradiction, assume that

$$||u(0)| - 1| > \lambda. \quad (4.24)$$

Then, by Lemma 2.1,

$$||u(x)| - 1| > \lambda/2 \text{ in } B_{\lambda/2K}. \quad (4.25)$$

By (4.23) and (4.25) and Lemma 2.2 we obtain

$$\frac{1}{4} \left(\frac{\lambda}{2}\right)^2 \left(\frac{\lambda}{2K}\right)^3 |B_1| \leq E(u; B_{\lambda/2K}) \leq E(u; B_1) \leq E(\tilde{R}_0)/\tilde{R}_0 \leq \tilde{C} \tilde{R}_0^{\alpha-1} \ln \tilde{R}_0. \quad (4.26)$$

Let $T = T(\lambda, \alpha)$ be a large enough value of \tilde{R}_0 for which (4.26) is violated, i.e.,

$$\frac{\lambda^5 |B_1|}{128K^3} > \tilde{C} T^{\alpha-1} \ln T. \quad (4.27)$$

Therefore, taking $\tilde{R}_0 = \tilde{R}_0(\gamma, \lambda)$ satisfying both (4.18) and $\tilde{R}_0 \geq T$, and then setting $R_1(\gamma, \lambda) := \tilde{R}_1(\tilde{R}_0)$ (see (4.22)), we see that (4.26), and hence also (4.24), cannot hold for $R \geq R_1(\gamma, \lambda)$. \square

Proof of Theorem 1.1. Fix any point $x \in \mathbb{R}^3$. For each $\lambda > 0$ we may apply Theorem 4.1 on $B_{R_n}(x)$ for an appropriate sequence $R_n \rightarrow \infty$ to conclude that $|u(x)| = 1$. Hence $|u| \equiv 1$ in \mathbb{R}^3 and from (1.1) we deduce that both components of $u = (u_1, u_2)$ are harmonic functions. Assuming without loss of generality that $u(0) = (1, 0)$, we conclude from the maximum principle that $u_1 \equiv 1$ and therefore $u \equiv (1, 0)$ in \mathbb{R}^3 . \square

Appendix. Proof of Lemma 3.3

Proof. We let $\varepsilon = 1/R$ and $v(x) = u(Rx)$. Then $v : S_1 \rightarrow \mathbb{R}^2$ satisfies

$$E_\varepsilon^{(T)}(v; S_1) \leq \gamma \ln \frac{1}{\varepsilon}. \quad (A.1)$$

Following [19], proof of Theorem 5.3, given $\delta > 0$ there exists a collection of disjoint spherical discs that we denote \mathcal{D}_0 such that each disc in the family has radius bounded below by $\Lambda\varepsilon$, such that $|v| > 1 - \delta$ on the complement of \mathcal{D}_0 and such that, denoting by r_0 the sum of the radii of the discs, we have

$$r_0 \leq C\gamma |\ln \varepsilon| \frac{\varepsilon}{\delta^3}. \quad (A.2)$$

It should be noted that in contrast with the situation considered in [19], the restriction of v to S_1 does not necessarily satisfy the Ginzburg-Landau equation (which would involve the Laplace-Beltrami operator on S_1). However, (2.2) implies that the restriction of v to S_1 satisfies an estimate of the form $\|\nabla v\|_{L^\infty} \leq C/\varepsilon$, which is what needed to construct \mathcal{D}_0 using the method of [19]. Then we apply the ball-growth procedure as in [19, Thm 4.2]. This yields, for any $t > 0$, a family of spherical discs $\mathcal{D}(t)$ which is increasing and such that the sum of the radii of the discs in $\mathcal{D}(t)$ is $e^t r_0$. Moreover, denoting

$$\mathcal{F}(x, r) = E_\varepsilon^{(T)}(v; \tilde{D}_r(x)) \quad \text{and} \quad \mathcal{F}(\mathcal{D}(t)) = \sum_{\tilde{D}_r(x) \in \mathcal{D}(t)} \mathcal{F}(x, r),$$

we deduce from [19, Proposition 4.1] and (A.1) that, for all $s > 0$,

$$\gamma |\ln \varepsilon| \geq \mathcal{F}(\mathcal{D}(s)) - \mathcal{F}(\mathcal{D}(0)) \geq \int_0^s \sum_{\tilde{D}_r(x) \in \mathcal{D}(t)} r \frac{\partial \mathcal{F}}{\partial r}(x, r) dt.$$

Hence, by Fubini's theorem there exists $t \in (0, s)$ such that $\mathcal{D}(t) = \{\tilde{D}_{\rho_i} = \tilde{D}_{\rho_i}(x_i), 1 \leq i \leq k\}$ satisfies

$$\sum_{i=1}^k \rho_i \frac{\partial \mathcal{F}}{\partial r}(x_i, \rho_i) \leq \frac{\gamma |\ln \varepsilon|}{s}.$$

Choosing $s = \frac{2\pi+\gamma}{4\pi} |\ln \varepsilon|$, we find that,

$$\sum_{i=1}^k \rho_i \frac{\partial \mathcal{F}}{\partial r}(x_i, \rho_i) \leq 2\pi \left(\frac{2\gamma}{\gamma + 2\pi} \right). \quad (\text{A.3})$$

On the other hand, since $|v| > 1 - \delta$ on the complement of \mathcal{D}_0 , we have for each i

$$\frac{\partial \mathcal{F}}{\partial r}(x_i, \rho_i) = E_\varepsilon^{(T)}(v; \partial \tilde{D}_{\rho_i}) \geq \frac{\pi}{\sin \rho_i} \deg(v, \partial \tilde{D}_{\rho_i})^2 (1 - \delta)^2 \geq \frac{\pi}{\rho_i} \deg(v, \partial \tilde{D}_{\rho_i})^2 (1 - \delta)^2,$$

where we used the fact that the Euclidean radius of the circle $\partial \tilde{D}_{\rho_i}$ is $\sin \rho_i$. Substituting in (A.3) yields

$$\sum_{i=1}^k \deg(v, \partial \tilde{D}_{\rho_i})^2 \leq \frac{2}{(1 - \delta)^2} \cdot \frac{2\gamma}{\gamma + 2\pi} < 2, \quad (\text{A.4})$$

the last inequality on the R.H.S. of (A.4) holds for $\delta > 0$ small enough, $\delta < \delta_0(\gamma)$. We thus fix any $\delta \in (0, \min(1/8, \delta_0(\gamma)))$. Since $\sum_{i=1}^k \deg(v, \partial \tilde{D}_{\rho_i}) = 0$, it follows from (A.4) that $\deg(v, \partial \tilde{D}_{\rho_i}) = 0$ for all i . Indeed, otherwise we would have at least two nonzero degrees, clearly violating (A.4).

Now the sum of the radii of the discs in $\{\tilde{D}_{\rho_i}\}_{i=1}^k$ is $r = e^t r_0$ hence bounded by $e^s r_0$. Using (A.2) we conclude that for some $C > 0$ depending on γ ,

$$r \leq C |\ln \varepsilon| \varepsilon \times \varepsilon^{-\frac{2\pi+\gamma}{4\pi}} = C |\ln \varepsilon| \varepsilon^{\frac{2\pi-\gamma}{4\pi}} \leq \varepsilon^{\tilde{\alpha}},$$

for every small enough ε , provided $0 < \tilde{\alpha} < \frac{2\pi-\gamma}{4\pi}$. We fix such a value of $\tilde{\alpha}$.

The family of spherical discs $\{\tilde{D}_{\rho_i}\}_{i=1}^k$ is therefore such that $|v| > 1/2$ outside the discs, their total radius is less than $\varepsilon^{\tilde{\alpha}}$, the winding number of v on the boundary of each disc is zero, and such that for each $1 \leq i \leq k$ we have using (A.3) that

$$E_\varepsilon^{(T)}(v; \partial \tilde{D}_{\rho_i}) \leq \frac{2\pi}{\rho_i} \frac{2\gamma}{\gamma + 2\pi} < \frac{2\pi}{\rho_i}.$$

Rescaling back, we see easily that the spherical discs on S_R , $\tilde{D}_{r_i}^R(Rx_i) := R\tilde{D}_{\rho_i}$, $i = 1, \dots, k$, satisfy all the assertions (3.32)–(3.36) of Lemma 3.3 with $\alpha = 1 - \tilde{\alpha}$. \square

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